Conformal Einstein equations and Cartan conformal connection

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Abstract

Necessary and sufficient conditions for a space-time to be conformal to an Einstein space-time are interpreted in terms of curvature restrictions for the corresponding Cartan conformal connection.

In Ref.[1] we gave necessary and sufficient conditions for a 4-dimensional metric to be conformal to an Einstein metric. One of these conditions, the vanishing of the Bach tensor of the metric, has been discussed by many authors [1, 2, 3, 4]. In particular, it was interpreted as being equivalent to the vanishing of the Yang-Mills current of the corresponding Cartan conformal connection. The other condition, which is given in terms of rather complicated equation on the Weyl tensor of the metric has not been analyzed from the point of view of the corresponding Cartan conformal connection. The purpose of this letter is to fill this gap.

Let M be a 4-dimensional manifold equipped with the conformal class of metrics $[g]$. Here we will be assuming that g has Lorentzian signature, but our results are also valid in the other two signatures.

Given a conformal class $[g]$ on M we choose a representative g for the metric. Let θ^μ, $\mu = 1, 2, 3, 4$, be a null (or orthonormal) coframe for g on M. This, in particular, means that $g = \eta_{\mu\nu}\theta^\mu\theta^\nu$, with all the coefficients $\eta_{\mu\nu}$ being constants. We define $\eta^{\mu\nu}$ by $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu{}_\rho$ and we will use $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to rise and lower the Greek indices, respectively. The metric

$$g' = \mathrm{e}^{-2\phi} g \tag{1}$$

conformally related to g will be represented by a coframe

$$\theta'^\mu = \mathrm{e}^{-\phi}\theta^\mu \tag{2}$$

so that

$$g' = \eta_{\mu\nu}\theta'^\mu\theta'^\nu \tag{3}$$

with the same $\eta_{\mu\nu}$ as in the expression (1). Given g and θ^μ we consider the 1-forms $\Gamma^\mu{}_\nu$ uniquely determined on M by the equations

$$\begin{gathered}\mathrm{d}\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0,\\ \Gamma_{\mu\nu} = \Gamma_{[\mu\nu]}, \qquad \text{where} \qquad \Gamma_{\mu\nu} = \eta_{\mu\rho}\Gamma^\rho{}_\nu.\end{gathered} \tag{4}$$

Using $\Gamma^\mu{}_\nu$ we calculate the Riemann tensor 2-forms

$$\Omega^\mu{}_\nu = \frac{1}{2}\Omega^\mu{}_{\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma = \mathrm{d}\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu, \tag{5}$$

and the Ricci part of the Riemann tensor

$$K_{\nu\sigma} = \Omega^\mu{}_{\nu\mu\sigma}, \qquad K = \eta^{\nu\sigma}K_{\nu\sigma} \qquad \text{and} \qquad S_{\nu\sigma} = K_{\nu\sigma} - \frac{1}{4}K\eta_{\nu\sigma}. \tag{6}$$

We recall that the metric g is called *Einstein* iff

$$S_{\mu\nu} = 0 \tag{7}$$

and that it is called *conformal to Einstein* iff there exists ϕ such that the metric $g' = \mathrm{e}^{-2\phi} g$ is Einstein.

In the following we will also need 1-forms

$$\tau_\nu = \left(-\tfrac{1}{2}S_{\nu\rho} - \tfrac{1}{24}K\eta_{\nu\rho}\right)\theta^\rho \tag{8}$$

and 2-forms

$$C^\mu{}_\nu = \mathrm{d}\Gamma^\mu{}_\nu + \theta^\mu \wedge \tau_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu + \tau^\mu \wedge \theta_\nu, \qquad \tau^\mu = \eta^{\mu\nu}\tau_\nu. \tag{9}$$

It follows that the 2-forms

$$C^{\mu}{}_{\nu} = \frac{1}{2} C^{\mu}{}_{\nu\rho\sigma} \theta^{\rho} \wedge \theta^{\sigma} \tag{10}$$

are the Weyl 2-forms associated with the Weyl tensor $C^{\mu}{}_{\nu\rho\sigma}$ of the metric g. It is known that the Weyl tensor obeys the following identity

$$C_{\alpha\mu\nu\rho} C^{\beta\mu\nu\rho} = \frac{1}{4} C^2 \delta^{\beta}{}_{\alpha}, \tag{11}$$

where

$$C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}. \tag{12}$$

Using the Bianchi identities one shows that the tensor $\tau_{\mu\nu\rho}$ defined on M via

$$D\tau_{\mu} = \frac{1}{2} \tau_{\mu\nu\rho} \theta^{\nu} \wedge \theta^{\rho} = \mathrm{d}\tau_{\mu} + \tau_{\nu} \wedge \Gamma^{\nu}{}_{\mu} \tag{13}$$

is

$$\tau_{\nu\rho\sigma} = \nabla_{\mu} C^{\mu}{}_{\nu\rho\sigma}, \tag{14}$$

where ∇_{μ} is the covariant derivative operator associated with D.

In the following we will also need the Bach tensor

$$B_{\mu\nu} = \nabla^{\rho} \nabla^{\sigma} C_{\mu\rho\nu\sigma} + \frac{1}{2} C_{\mu\rho\nu\sigma} K^{\rho\sigma} \tag{15}$$

and a tensor

$$N_{\nu\rho\sigma} = (\nabla_{\alpha} C^{\alpha}{}_{\beta\gamma\delta}) C^{\mu\beta\gamma\delta} C_{\mu\nu\rho\sigma} - \frac{1}{4} C^2 \nabla_{\mu} C^{\mu}{}_{\nu\rho\sigma}. \tag{16}$$

It is known that the vanishing of the Bach tensor is a conformally invariant property. If $C^2 \neq 0$ the vanishing of $N_{\mu\nu\rho}$ is also conformally invariant. The relevance of both these tensors in the context of this letter is given by the following theorem [1].

Theorem 1 *Assume that the metric g satisfies the genericity condition*

$$C^2 \neq 0$$

on M. Then the metric is locally conformally equivalent to the Einstein metric if and only if

$$\text{(i)} \quad B_{\mu\nu} = 0 \quad \text{and} \quad \text{(ii)} \quad N_{\nu\rho\sigma} = 0 \tag{17}$$

on M.

Remarks

- Conditions (i) and (ii) are independent. In particular, metrics with vanishing Bach tensor and not conformal to Einstein metrics are known [5].
- If $C^2 = 0$ the condition (ii) must be replaced by another condition for the above theorem to be true. This another condition depends on the algebraic type of the Weyl tensor and is given in [1].
- Baston and Mason [3] gave another version of the above theorem in which condition (ii) was replaced by the vanishing of a different tensor than $N_{\nu\rho\sigma}$. Unlike $N_{\nu\rho\sigma}$, which is *cubic* in the Weyl tensor, the Baston-Mason tensor $E_{\nu\rho\sigma}$, is only *quadratic* in $C^{\mu}{}_{\nu\rho\sigma}$.

- Merkulov [4] interpreted condition (ii) as the vanishing of the Yang-Mills current of the *Cartan normal conformal connection* ω associated with the metric g. Following him, Baston and Mason [3] interpreted the condition $E_{\nu\rho\sigma}=0$ in terms of curvature condition for ω.

Although in the context of Theorem 1 conditions (ii) and $E_{\nu\rho\sigma}=0$ are equivalent, the tensors $N_{\mu\nu\rho}$ and $E_{\mu\nu\rho}$ are quite different. In addition to cubic versus quadratic dependence on the Weyl tensor, one can mention the fact that it is quite easy to express tensor $E_{\nu\rho\sigma}$ in the spinorial language and quite complicated in the tensorial language. Totally oposite situation occurs for the tensor $N_{\nu\rho\sigma}$. One of the motivation for the present letter is the existence of the normal conformal connection interpretation for the condition $E_{\nu\rho\sigma}=0$. As far as we know such interpretation of $N_{\nu\rho\sigma}=0$ has not been discussed. To fill this gap we first give the formal definition of the Cartan normal conformal connection. In order to do this we first, introduce the 6×6 matrix

$$Q_{AB}=\begin{pmatrix}0 & 0 & -1\\ 0 & \eta_{\mu\nu} & 0\\ -1 & 0 & 0\end{pmatrix} \tag{18}$$

and then define the $\mathbf{so}(2,4)$-valued 1-form $\tilde{\omega}$ on M by

$$\tilde{\omega}=\begin{pmatrix}0 & \tau_\mu & 0\\ \theta^\nu & \Gamma^\nu{}_\mu & \eta^{\nu\rho}\tau_\rho\\ 0 & \eta_{\mu\rho}\theta^\rho & 0\end{pmatrix}. \tag{19}$$

Then, we use a Lie subgroup $\mathbf{H}$ of $\mathbf{SO}(2,4)$, generated by the 6×6 matrices of the form

$$b=\begin{pmatrix}\mathrm{e}^{-\phi} & \mathrm{e}^{-\phi}\xi_\mu & \frac{1}{2}\mathrm{e}^{-\phi}\xi_\mu\xi_\nu\eta^{\mu\nu}\\ 0 & \Lambda^\nu{}_\mu & \Lambda^\nu{}_\rho\eta^{\rho\sigma}\xi_\sigma\\ 0 & 0 & \mathrm{e}^{\phi}\end{pmatrix},\qquad \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\eta_{\mu\nu}=\eta_{\rho\sigma}, \tag{20}$$

to lift the form $\tilde{\omega}$ to an $\mathbf{so}(2,4)$-valued 1-form ω on $M\times H$. Explicitely, if b is a generic element of $\mathbf{H}$, we put

$$\omega=b^{-1}\tilde{\omega}b+b^{-1}\mathrm{d}b,$$

so that

$$\omega=\begin{pmatrix}-\frac{1}{2}A & \tau'_\mu & 0\\ \theta'^\nu & \Gamma'^\nu{}_\mu & \eta^{\nu\sigma}\tau'_\sigma\\ 0 & \theta'^\sigma\eta_{\sigma\mu} & \frac{1}{2}A\end{pmatrix}, \tag{21}$$

with

$$\theta'^\nu=\mathrm{e}^{-\phi}\Lambda^{-1\nu}{}_\rho\theta^\rho, \tag{22}$$

$$A=2\xi_\mu\theta'^\mu+2\mathrm{d}\phi,$$

$$\Gamma'^\nu{}_\mu=\Lambda^{-1\nu}{}_\rho\Gamma^\rho{}_\sigma\Lambda^\sigma{}_\mu+\Lambda^{-1\nu}{}_\rho\mathrm{d}\Lambda^\rho{}_\mu+\theta'^\nu\xi_\mu-\xi^\nu\eta_{\mu\rho}\theta'^\rho,$$

$$\tau'_\mu=\mathrm{e}^{\phi}\tau_\nu\Lambda^\nu{}_\mu-\xi_\alpha\Lambda^{-1\alpha}{}_\rho\Gamma^\rho{}_\nu\Lambda^\nu{}_\mu+\frac{1}{2}\mathrm{e}^{-\phi}\eta^{\alpha\beta}\xi_\alpha\xi_\beta\tau_\nu\Lambda^\nu{}_\mu-\frac{1}{2}\xi_\mu A+\mathrm{d}\xi_\mu-\xi_\alpha\Lambda^{-1\alpha}{}_\rho\mathrm{d}\Lambda^\rho{}_\mu$$

The so defined form ω is the Cartan normal conformal connection associated with the conformal class $[g]$ written in a particular trivialization of an approppriately defined H-bundle over M. It can be viewed as a useful tool for encoding conformal properties of the metrics on manifolds. Indeed, if given g on M one calculates the quantities $\theta^\mu, \Gamma^\mu{}_\nu, \tau_\mu$, then the corresponding quantities for the conformally rescaled metric $g' = \mathrm{e}^{-2\phi} g$ are given by (21) with $A = 0$.[1]

The curvature of ω is

$$R = \mathrm{d}\omega + \omega \wedge \omega$$

and has the rather simple form

$$R = \begin{pmatrix} 0 & (D\tau_\mu)' & 0 \\ 0 & C'^\nu{}_\mu & g^{\nu\mu}(D\tau_\mu)' \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

with the 2-forms $C'^\nu{}_\mu$ and $(D\tau_\mu)'$ defined by

$$\begin{aligned} C'^\nu{}_\mu &= \Lambda^{-1\nu}{}_\rho C^\rho{}_\sigma \Lambda^\sigma{}_\mu \\ (D\tau_\mu)' &= \mathrm{e}^\phi \mathrm{D}\tau_\nu \Lambda^\nu{}_\mu - \xi_\alpha \Lambda^{-1\alpha}{}_\rho \mathbf{C}^\rho{}_\nu \Lambda^\nu{}_\mu. \end{aligned} \tag{24}$$

Similarly to the properties of ω, the curvature R can be used to extract the transformations of $D\tau_\mu$ and $C^\nu{}_\mu$ under the conformal rescaling of the metrics. If $g \to g' = \mathrm{e}^{-2\phi} g$ these transformations are given by (24) with $\xi_\mu = -\nabla'_\mu \phi$. In particular, if we freeze the Lorentz transformations of the tetrad, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$, then we see that the Weyl 2-forms $C^\nu{}_\mu$ constitute the conformal invariant.

The curvature R of the Cartan normal connection ω is horizontal which, in other words, means that it has only $\theta^\mu \wedge \theta^\nu$ terms in the decomposition onto the basis of forms $(\theta^\mu, \mathrm{d}\phi, \Lambda^{-1\mu}{}_\nu \mathrm{d}\Lambda^\nu{}_\rho, \mathrm{d}\xi_\mu)$. Thus, the Hodge $*$ operator associated with g on M is well defined acting on R and in consequence the Yang-Mills equations for R can be written

$$D * R = \mathrm{d} * R - *R \wedge \omega + \omega \wedge *R = 0. \tag{25}$$

The following theorem is well known (see e.g. [2]).

Theorem 2 *The metric g on a 4-dimansional manifold M satisfies the Bach equations $B_{\mu\nu} = 0$ if and only if it satisfies the Yang-Mills equations $D * R = 0$ for the Cartan normal conformal connection associated with g.*

In view of this theorem it is natural to ask about the normal conformal connection interpretation of $N_{\mu\nu\rho} = 0$, which together with $B_{\mu\nu} = 0$ are sufficient for g to be conformal to Einstein. To answer this question we introduce indices $A, B, C, ...$ which run from 0 to 5 and attach them to any 6×6 matrix. In this way the elements of matrix R are 2-forms

$$R^A{}_B = \frac{1}{2} R^A{}_{B\mu\nu} \theta'^\mu \wedge \theta'^\nu. \tag{26}$$

We define the 6×6 matrix of 2-forms $\hat{R}^3$, which is the appropriately contracted triple product of R, with matrix elements

$$\hat{R}^3_{AF} = \frac{1}{2} Q_{AE} R^E{}_{B\alpha\beta} R^B{}_{C\gamma\delta} R^C{}_{F\mu\nu} \eta^{\alpha\gamma} \eta^{\beta\delta} \theta'^\mu \wedge \theta'^\nu. \tag{27}$$

[1] Note that $A = 0$ means that $\xi_\mu = -\nabla'_\mu \phi$, where $\nabla'_\mu = e^\phi \Lambda^\nu{}_\mu \nabla_\nu$. Admitting ξ_μ which are not gradients in the definition of $\mathbf{H}$ allows for transformations between different Weyl geometries $[(g, A)]$

The symmetric part of this matrix $(\hat{R}^3)^T + \hat{R}^3$ is of the form

$$(\hat{R}^3)^T + \hat{R}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P_\mu \\ 0 & P_\mu & P \end{pmatrix} \tag{28}$$

where P_μ and P are appropriate 2-forms on $M \times H$. It follows that under the assumption that

$$C^2 \neq 0 \tag{29}$$

this matrix has particularly simple form

$$(\hat{R}^3)^T + \hat{R}^3 = \frac{1}{2}\mathrm{e}^{6\phi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\phi}\Lambda^\alpha{}_\sigma N_{\alpha\beta\gamma}\theta^\beta \wedge \theta^\gamma \\ 0 & \mathrm{e}^{-\phi}\Lambda^\alpha{}_\sigma N_{\alpha\beta\gamma}\theta^\beta \wedge \theta^\gamma & V^\alpha N_{\alpha\beta\gamma}\theta^\beta \wedge \theta^\gamma \end{pmatrix} \tag{30}$$

where

$$V^\alpha = \frac{4}{C^2}(\nabla_\rho C^\rho{}_{\lambda\tau\sigma})C^{\alpha\lambda\tau\sigma} - \mathrm{e}^{-\phi}\xi_\rho \Lambda^{-1\rho}{}_\lambda \eta^{\lambda\alpha} \tag{31}$$

and $N_{\alpha\beta\gamma}$ is given by (16). The proof of this fact consists of a straightforward but lengthy calculation which uses the identities (11) and (14). This enables us to formulate the following theorem.

Theorem 3 *Assume that the metric g satisfies*

$$C^2 \neq 0$$

on M. Let ω be its Cartan normal conformal connection with curvature R and the matrix $\hat{R}^3$ as above. Then the metric is locally conformally equivalent to the Einstein metric if and only if the

$$\text{(i)} \quad D * R = 0 \qquad \text{and} \qquad \text{(ii)} \quad (\hat{R}^3)^T + \hat{R}^3 = 0. \tag{32}$$

The above condition (ii) can now be compared to the Baston-Mason conformal connection interpretation of the condition $E_{\mu\nu\rho} = 0$. According to them, this condition is [3]

$$\text{(ii}'\text{)} \qquad [R^+_{\mu\nu}, R^-_{\rho\sigma}] = 0, \tag{33}$$

where $R^+ = \frac{1}{2}R^+_{\mu\nu}\theta^\mu \wedge \theta^\nu$ and $R^- = \frac{1}{2}R^-_{\mu\nu}\theta^\mu \wedge \theta^\nu$ denote, respectively, the self-dual and anti-self-dual parts of the curvature R, i.e. $*R^\pm = \pm iR^\pm$, and $R = R^+ \oplus R^-$.

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